TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 2, August 1983

SUPPORT POINTS OF FAMILIES OF ANALYTIC FUNCTIONS DESCRIBED BY SUBORDINATION

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ABSTRACT. We determine the set of support points for several families of functions analytic in the open unit disc and which are generally defined in terms of subordination. The families we study include the functions with a positive real part, the typically-real functions, and the functions which are subordinate to a given majorant. If the majorant F is univalent then each support point has the form $F \circ \phi$, where ϕ is a finite Blaschke product and $\phi(0) = 0$. This completely characterizes the set of support points when F is convex. The set of support points is found for some specific majorants, including $F(z) = ((1+z)/(1-z))^p$ where $p > 1$. Let K and St denote the set of normalized convex and starlike mappings, respectively. We find the support points of the families K^* and St* defined by the property of being subordinate to some member of K or St, respectively.

Introduction. Let $\Delta = \{z \in \mathbf{C}: |z| < 1\}$ and let $\mathcal{Q}$ denote the set of functions analytic in Δ. Then $\mathcal{Q}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. By a continuous, linear functional on $\mathcal{Q}$ we mean a complex-valued functional defined on $\mathcal{Q}$ that is linear and continuous. In other words, if J is such a functional then $J(af + bg) = aJ(f) + bJ(g)$ whenever a and b belong to $\mathbf{C}$ and f and g belong to $\mathcal{Q}$. Also, if $f_n \in \mathcal{Q}$ $(n = 1, 2, \ldots)$ and $f_n \to f$, then $J(f_n) \to J(f)$. Each continuous, linear functional J on $\mathcal{Q}$ is given by a sequence $\{b_n\}$ $(n = 0, 1, \ldots)$ which satisfies $\overline{\lim}_{n\to\infty} \sqrt[n]{|b_n|} < 1$ and

$$J(f) = \sum_{n=0}^{\infty} a_n b_n \quad \text{where} \, f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1) \qquad [\mathbf{13}].$$

For such a sequence $\{b_n\}$, the function $F(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $\overline{\Delta} = \{z \in \mathbf{C}: |z| \leq 1\}$. We shall use the notation J_F in this context.

A function f is called a support point of a compact subset $\mathcal{F}$ of $\mathcal{Q}$ if $f \in \mathcal{F}$ and if there is a continuous, linear functional J on $\mathcal{Q}$ so that

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g): g \in \mathcal{F}\}$$

Received by the editors May 24, 1982.
1980 *Mathematics Subject Classification*. Primary 30C45, 30C80, 30D50; Secondary 30C70, 30C75.
Key words and phrases. Analytic function, continuous linear functional, support point, extreme point, closed convex hull, subordination, univalent function, convex mapping, starlike mapping, function with a positive real part, bounded function, Herglotz formula, probability measure, finite Blaschke product.
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0002-9947/82/0000-1018/$07.50

and $\operatorname{Re} J$ is nonconstant on $\mathfrak{F}$. In general, for a fixed continuous, linear functional J the solution set contains an extreme point of $\mathfrak{H}\mathfrak{F}$, the closed convex hull of $\mathfrak{F}$. If there is a unique solution then that function belongs to $\mathfrak{E}\mathfrak{H}\mathfrak{F}$, the set of extreme points of $\mathfrak{H}\mathfrak{F}$. We shall denote the set of support points of $\mathfrak{F}$ by $\operatorname{supp} \mathfrak{F}$.

We recall the definition of subordination between two functions, say f and F, analytic in Δ. This means that there is an analytic function ϕ so that $\phi(0) = 0$, $|\phi(z)| < 1$ and $f(z) = F(\phi(z))$ for $|z| < 1$. This relation shall be denoted by $f \prec F$. If F is univalent in Δ the subordination is equivalent to $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$. In [**7**, **9**] the relation between extreme-point theory and subordination is explored.

Let S denote the set of functions that are analytic and univalent in Δ and satisfy $f(0) = 0$ and $f'(0) = 1$, and let K and St denote the subsets of S for which $f(\Delta)$ is convex or starlike, respectively. Also, let $K^* = \{f\colon f \prec g \text{ for some } g \text{ in } K\}$ and $\mathrm{St}^* = \{f\colon f \prec g \text{ for some } g \text{ in St}\}$. In [**7**, pp. 458–459] it was proven that

$$\mathfrak{E}\mathfrak{H}K^* = \left\{\frac{xz}{1 - yz} : |x| = |y| = 1\right\} \quad \text{and} \quad \mathfrak{E}\mathfrak{H}\mathrm{St}^* = \left\{\frac{xz}{(1 - yz)^2} : |x| = |y| = 1\right\}.$$

Let $\mathfrak{P}$ denote the set of functions p that are analytic in Δ and satisfy $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $|z| < 1$. Let $\mathfrak{B}$ denote the set of functions ϕ that are analytic in Δ and satisfy $|\phi(z)| \leqslant 1$ for $|z| < 1$, and let $\mathfrak{B}_0$ denote the subset of $\mathfrak{B}$ given by the additional condition $\phi(0) = 0$. Then, $p \in \mathfrak{P}$ if and only if $p = (1 + \phi)/(1 - \phi)$ for some ϕ in $\mathfrak{B}_0$.

Let F be a nonconstant, analytic function in Δ and let $\mathfrak{F}$ be the family of functions subordinate to F in Δ. In [**7**, p. 463] we proved that each function $F(xz)$, $|x| = 1$, belongs to both $\mathfrak{E}\mathfrak{H}\mathfrak{F}$ and $\operatorname{supp} \mathfrak{F}$.

In this paper we examine $\operatorname{supp} \mathfrak{F}$ for a number of majorants F, including cases where $\mathfrak{E}\mathfrak{H}\mathfrak{F}$ is relatively small and relatively numberous. In particular, we consider $\mathfrak{P}$ and $\mathfrak{B}_0$, defined by the majorants $F(z) = (1 + z)/(1 - z)$ and $F(z) = z$, respectively. We note that $\mathfrak{E}\mathfrak{P} = \{(1 + xz)/(1 - xz)\colon |x| = 1\}$ and $\mathfrak{E}\mathfrak{B}_0$ is quite diverse [**8**, p. 138].

In §1 we determine the support points of $\mathfrak{P}$ and of some related families, including the typically-real functions. In §2 we prove that if $F'(z) \neq 0$ ($|z| < 1$), then $\operatorname{supp} \mathfrak{F}$ is contained in the set $\{F \circ \phi\}$, where ϕ is a finite Blaschke product and $\phi(0) = 0$. In the case that F is convex, $\operatorname{supp} \mathfrak{F}$ is completely determined. This result leads to the fact that for such a family $\mathfrak{F}$ with $F(\Delta)$ not a half-plane, there is a unique solution to each linear extremal problem, and thus $\operatorname{supp} \mathfrak{F} \subset \mathfrak{E}\mathfrak{F}$. For a convex mapping F, where $F(\Delta)$ is not a half-plane, $\mathfrak{E}\mathfrak{F}$ is quite varied [**1**] and so $\operatorname{supp} \mathfrak{F}$ is much smaller than $\mathfrak{E}\mathfrak{F}$.

In §3 we consider the case of $F(z) = ((1 + cz)/(1 - z))^p$ where $|c| \leqslant 1$, $c \neq -1$ and $p > 1$. For this majorant, it is known that

$$\mathfrak{E}\mathfrak{H}\mathfrak{F} = \left\{((1 + cxz)/(1 - xz))^p : |x| = 1\right\} \text{ [2]}.$$

We prove that $\operatorname{supp} \mathfrak{F} = \mathfrak{E}\mathfrak{H}\mathfrak{F}$, which contrasts with the results for $\operatorname{supp} \mathfrak{P}$, corresponding to the case $p = 1$ and $c = 1$. We also treat some related families, for example, the set of functions that are subordinate to $F(z) = ((1 + z)/(1 - z))^p$ ($p > 1$) and are real on the real axis.

In [3] it was shown that

$$\operatorname{supp} K = \mathfrak{E}\mathfrak{H}K = \{z/(1-xz): |x| = 1\}$$

and

$$\operatorname{supp} \mathrm{St} = \mathfrak{E}\mathfrak{H}\mathrm{St} = \{z/(1-xz)^2: |x| = 1\}.$$

In §4 we prove that supp St* consists of all functions of the form $f(xz)$ where $f \in \mathrm{St}$ and $|x| = 1$. However, if J does not have the form $J(f) = \alpha f(0) + \beta f'(0)$, then the support points of St* associated with J belong to $\mathfrak{E}\mathfrak{H}\mathrm{St}^*$. If J is given by $\{b_n\}$ and $F(z) = \sum_{n=0}^{\infty} b_n z^n$, then the support points of K^* associated with J are described in terms of whether or not F maps Δ onto a disc centered at $w = 0$. We find that supp K^* consists of all functions $G \circ \phi$ where $G \in K$ and ϕ is a finite Blaschke product with $\phi(0) = 0$.

Support points of $\mathcal{P}$ and $\mathfrak{B}$.

LEMMA 1. *Let J be a continuous, linear functional on $\mathcal{Q}$.* Re J *is constant on $\mathcal{P}$ if and only if J has the form*

$$J(f) = \alpha f(0) \tag{1}$$

where $f \in \mathcal{Q}$ and $\alpha \in \mathbf{C}$.

PROOF. If J is given by (1) and $p \in \mathcal{P}$, then $J(p) = \alpha p(0) = \alpha$ and, so, Re J is constant on $\mathcal{P}$.

Conversely, suppose that Re J is constant on $\mathcal{P}$. If $p(z) = 1 + xz^n$ where $|x| = 1$ and $n = 1, 2, \ldots$, then $p \in \mathcal{P}$. Let J be given by the sequence $\{b_n\}$. Since Re J is constant on $\mathcal{P}$, $\operatorname{Re} J(1 + xz^n) = \operatorname{Re} b_0 + \operatorname{Re}(xb_n)$ is constant. With n fixed, this implies that $\operatorname{Re}(xb_n)$ is constant for $|x| = 1$ and, so, $b_n = 0$ for $n = 1, 2, \ldots$. Thus, $J(f) = b_0 a_0$ whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). Hence, J has the form (1).

THEOREM 1. *The set of support points of $\mathcal{P}$ consists of all functions which may be written*

$$p(z) = \sum_{k=1}^{m} \lambda_k \frac{1 + x_k z}{1 - x_k z}, \tag{2}$$

where $\lambda_k \geqslant 0$, $\sum_{k=1}^{m} \lambda_k = 1$ and $|x_k| = 1$ ($m = 1, 2, \ldots$).

PROOF. Suppose that $p_0 \in \operatorname{supp} \mathcal{P}$. There is a continuous, linear functional J: $\mathcal{Q} \to \mathbf{C}$ such that

$$\operatorname{Re} J(p_0) = \max\{\operatorname{Re} J(p): p \in \mathcal{P}\} \tag{3}$$

and Re J is not constant on $\mathcal{P}$. Let J be given by the sequence $\{b_n\}$ and let

$$P(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

Then

$$J(P(xz)) = b_0 + \sum_{n=1}^{\infty} 2b_n x^n = G(x)$$

defines a function which is analytic in $\bar{\Delta}$, since $\overline{\lim}_{n\to\infty}\sqrt[n]{|b_n|} < 1$. Since $\operatorname{Re} J$ is not constant on $\mathfrak{P}$, Lemma 1 implies that G is not constant (alternatively, if $\operatorname{Re} G$ is constant then $\operatorname{Re} J$ is constant on $\mathfrak{E}\mathfrak{P}$ and thus on $\mathfrak{P}$). Therefore, there are a finite number of distinct values of x [**3**, p. 106], say $x_1, x_2, \ldots, x_m$, so that

$$\operatorname{Re} G(x) = \max\{\operatorname{Re} G(y) \colon |y| = 1\}. \tag{4}$$

If $\mathfrak{G} = \{q \in \mathfrak{P}\colon \operatorname{Re} J(q) = \max_{p\in\mathfrak{P}} \operatorname{Re} J(p)\}$, then $\mathfrak{G}$ is compact, convex and nonvoid and, so, $\mathfrak{G}$ has extreme points. As $\mathfrak{G}$ is an extremal subset of $\mathfrak{P}$, $\mathfrak{E}\mathfrak{G} \subset \mathfrak{E}\mathfrak{P}$. Therefore, $\mathfrak{E}\mathfrak{G} = \{(1 + x_k z)/(1 - x_k z)\colon k = 1, 2, \ldots, m\}$ and so $\mathfrak{G}$ consists of the functions given by (2) [**3**, p. 100]. In particular, p_0 must have that form.

Conversely, suppose that p_0 has the form (2) where the x_k are distinct. There is a function G [**3**, p.101] that is analytic in $\bar{\Delta}$ so that (4) holds with $|x| = 1$ if and only if $x = x_k$ $(k = 1, 2, \ldots, m)$. If we let $G(x) = \sum_{n=0}^{\infty} B_n x^n$, $b_0 = B_0$ and $b_n = B_n/2$ for $n = 1, 2, \ldots,$ then $\overline{\lim}_{n\to\infty}\sqrt[n]{|b_n|} < 1$, and so $\{b_n\}$ defines a continuous, linear functional J on $\mathfrak{A}$. This shows that $p_0 \in \operatorname{supp} \mathfrak{P}$, since p_0 is in the solution set to (3) and $\operatorname{Re} J$ is not constant on $\mathfrak{P}$.

REMARKS. 1. Let $F(z) = (1 + cz)/(1 - z)$ $(c \in \mathbf{C}, c \neq -1)$ and let $\mathfrak{F} = \{f\colon f \prec F\}$. Since

$$\frac{1+cz}{1-z} = \frac{1-c}{2} + \frac{1+c}{2}\,\frac{1+z}{1-z},$$

it is clear that $\operatorname{supp} \mathfrak{F}$ consists of functions of the form

$$p(z) = \sum_{k=1}^{m} \lambda_k \frac{1 + c x_k z}{1 - x_k z}.$$

2. Theorem 1 implies that $\mathfrak{E}\mathfrak{P} \subsetneq \operatorname{supp} \mathfrak{P}$ and $\operatorname{supp} \mathfrak{P}$ is dense in $\mathfrak{P}$, since the functions (2), with $m = 1, 2, \ldots,$ form a dense subset of $\mathfrak{P}$.

3. [Theorem 1 may not be new. The referee suggests that it likely was proved earlier using a variational method of G. M. Golusin or one of M. S. Robertson.] The kind of argument we give in the proof of Theorem 1 was introduced in [**3**]. It is interesting to find that this result, in turn, may be used to obtain one direction of the next theorem concerning the support points of $\mathfrak{B}$. This result was first proved by P. C. Cochrane and the second author [**4**].

THEOREM 2. *The set of support points of $\mathfrak{B}$ consists of all finite Blaschke products, that is, functions of the form*

$$\phi(z) = x \prod_{k=1}^{m} \frac{z + \alpha_k}{1 + \bar{\alpha}_k z} \tag{5}$$

where $|x| = 1$ and $|\alpha_k| \leq 1$ $(m = 1, 2, \ldots)$.

We shall give a new proof that if $\phi_0 \in \operatorname{supp} \mathfrak{B}$ then ϕ is a finite Blaschke product. Suppose that $\phi_0 \in \operatorname{supp} \mathfrak{B}$. There is a continuous, linear functional $J\colon \mathfrak{A} \to \mathbf{C}$ so that

$$\operatorname{Re} J(\phi_0) = \max\{\operatorname{Re} J(\phi)\colon \phi \in \mathfrak{B}\} \tag{6}$$

and $\operatorname{Re} J$ is not constant on $\mathfrak{B}$. The relation

$$p(z) = \frac{1 + z\phi(z)}{1 - z\phi(z)} \tag{7}$$

or, equivalently,

$$\phi(z) = \frac{1}{z}\frac{p(z) - 1}{p(z) + 1} \tag{8}$$

defines a one-to-one correspondence between $\mathfrak{B}$ and $\mathfrak{P}$. Let

$$p_0(z) = \frac{1 + z\phi_0(z)}{1 - z\phi_0(z)}$$

and define a functional I: $\mathfrak{P} \to \mathbf{C}$ by $I(p) = J(\phi)$ where ϕ and p correspond by (7).

Suppose that $p, q \in \mathfrak{P}$ and $0 \leqslant \varepsilon \leqslant 1$. Then $(1 - \varepsilon)p + \varepsilon q \in \mathfrak{P}$ and

$$\begin{aligned} I[(1-\varepsilon)p + \varepsilon q] &= J\left[\frac{1}{z}\frac{p - 1 + \varepsilon(q-p)}{p + 1 + \varepsilon(q-p)}\right] \\ &= J\left[\frac{1}{z}\frac{p-1}{p+1} + \left\{\frac{2}{z}\frac{q-p}{(p+1)^2}\right\}\varepsilon + o(\varepsilon)\right] \\ &= J(\phi) + 2J\left[\frac{1}{z}\frac{q-p}{(p+1)^2}\right]\varepsilon + o(\varepsilon). \end{aligned}$$

(The first term $o(\varepsilon)$ indicates a function which is analytic in Δ and which, when divided by ε, tends to 0 uniformly on compact subsets of Δ, as $\varepsilon \to 0^+$.) In particular, by letting $\phi = \phi_0$ we conclude that

$$\operatorname{Re}\left\{2J\left[z^{-1}(q - p_0)/(p_0 + 1)^2\right]\varepsilon + o(\varepsilon)\right\} \leqslant 0. \tag{9}$$

Dividing (9) by $\varepsilon > 0$ and letting $\varepsilon \to 0$, we infer that

$$\operatorname{Re} J\left[\hat{q}/z(\hat{p}_0 + 2)^2\right] \leqslant \operatorname{Re} J\left[\hat{p}_0/z(\hat{p}_0 + 2)^2\right] \tag{10}$$

for all $\hat{q}$ in $\hat{\mathfrak{P}}$, where $\hat{\mathfrak{P}} = \mathfrak{P} - 1$, $\hat{q} = q - 1$ and $\hat{p}_0 = p_0 - 1$. The functional L, defined by $L(f) = J[f/z(\hat{p}_0 + 2)^2]$, is a continuous, linear functional on $\mathfrak{A}_0$, the subset of $\mathfrak{A}$ for which $f(0) = 0$. If $b_0 \in \mathbf{C}$ and $L^*(f) = b_0 f(0) + L[f - f(0)]$, then L^* extends L to $\mathfrak{A}$, and L^* is a continuous, linear functional on $\mathfrak{A}$.

We claim that $\operatorname{Re} L$ is not constant on $\hat{\mathfrak{P}}$. Suppose otherwise; then, as $p = 1$ belongs to $\mathfrak{P}$, we conclude that $\operatorname{Re} L(p) = 0$ for all p in $\hat{\mathfrak{P}}$. Since $1 + z^n \in \mathfrak{P}$ $(n = 1, 2, \ldots)$, $z^n \in \hat{\mathfrak{P}}$, and this implies that $\operatorname{Re} L(\alpha z^n) = 0$ whenever $\alpha \in \mathbf{C}$ and $n = 1, 2, \ldots$. Then we have $\operatorname{Re} L[\Sigma_{k=1}^N \alpha_k z^k] = 0$ whenever $\alpha_k \in \mathbf{C}$ and $N = 1, 2, \ldots$. The continuity of L implies that $\operatorname{Re} L(f) = 0$ whenever $f \in \mathfrak{A}_0$. In particular, this asserts that $\operatorname{Re} L(f) = 0$, where $f(z) = xz^{n+1}(\hat{p}_0 + 2)^2$ $(|x| = 1,\ n = 1, 2, \ldots)$, that is, $\operatorname{Re} J(xz^n) = 0$. If J is given by the sequence $\{b_n\}$ then this implies that $\operatorname{Re} xb_n = 0$. With n fixed, the relation $\operatorname{Re} xb_n = 0$ for $|x| = 1$ implies that $b_n = 0$. Therefore, $b_n = 0$ for $n = 1, 2, \ldots$, and so $J(f) = b_0 f(0)$.

Equation (10) and that fact that Re L^* is not constant on $\mathcal{P}$ imply that $p_0 \in$ supp $\mathcal{P}$. Theorem 1 implies that p_0 has the form (2), which implies that ϕ_0 is a finite Blaschke product [4, p. 83]. Therefore, ϕ_0 has the form (5) with m replaced by $m - 1$.

REMARKS. 1. It is clear from Theorem 2 that supp $\mathfrak{B}_0$ consists of all functions of the form

$$\phi(z) = xz \prod_{k=1}^{m} \frac{z + \alpha_k}{1 + \bar{\alpha}_k z} \tag{11}$$

where $|x| = 1$, $|\alpha_k| \leqslant 1$ and $m = 1, 2, \ldots$.

2. It is known [8, p. 138] that $\mathfrak{E}\mathfrak{B}_0$ consists of all functions f in $\mathfrak{B}_0$ which satisfy $\int_0^{2\pi} \log(1 - |f(e^{i\theta})|)\, d\theta = -\infty$, and so we see that supp $\mathfrak{B}_0 \subset \mathfrak{E}\mathfrak{B}_0$, but supp $\mathfrak{B}_0$ is a much more restricted set than $\mathfrak{E}\mathfrak{B}_0$.

3. Every function on the boundary of the unit ball of H^p is known to be an extreme point of the ball, whenever $p > 1$. Using the Cauchy-Schwarz inequality it is easy to prove that the set of support points of the unit ball of H^2 consists of the functions on the boundary of the ball which, in addition, are analytic in $\bar{\Delta}$. This provides another example where the set of support points is must more restricted than the set of extreme points. We ask the question: Does a similar result hold for the unit ball of H^p whenever $1 < p < \infty$?

4. For a key idea used in the proof of Theorem 2 we are indebted to Stephen D. Fisher. In an unpublished paper, Fisher introduced the technique of regarding $(1 - \varepsilon)p + \varepsilon q = p + (q - p)\varepsilon$ as a variation of p, whenever p and q belong to a convex set. The use of this variation is considerably simpler than the variation on which the proof of one direction of Theorem 2 given in [4] ultimately depends. We again use Fisher's idea in the proof of Theorem 4. The proof that each Blaschke product (5) belongs to supp $\mathfrak{B}$, given in [4], depends on a careful examination of the Schur algorithm.

We shall determine the support points of the family, denoted T, of typically-real functions introduced by W. Rogosinski [10]. First we treat a related family. Namely, let $\mathcal{P}_{\mathbf{R}}$ denote the subset of $\mathcal{P}$ of functions satisfying $p(z)$ is real when z is real $(-1 < z < 1)$. $\mathfrak{E}\mathcal{P}_{\mathbf{R}}$ consists of all functions $p(z) = (1 - z^2)/(1 - xz)(1 - \bar{x}z)$ where $|x| = 1$ and Im $x \geqslant 0$. If J is a continuous, linear functional on $\mathcal{Q}$ such that Re J is not constant on $\mathcal{P}_{\mathbf{R}}$, then J operating on $\mathfrak{E}\mathcal{P}_{\mathbf{R}}$ defines an analytic function on $\partial\Delta = \{z: |z| = 1\}$. With an argument similar to that in Theorem 1, this leads to the result that each support point of $\mathcal{P}_{\mathbf{R}}$ has the form

$$p(z) = \sum_{k=1}^{m} \lambda_k \frac{1 - z^2}{(1 - x_k z)(1 - \bar{x}_k z)},$$

where $\lambda_k \geqslant 0$, $\sum_{k=1}^{m} \lambda_k = 1$, $|x_k| = 1$ and Im $x_k \geqslant 0$. Part of this argument requires showing that Re J is constant on $\mathcal{P}_{\mathbf{R}}$ if and only if the sequence $\{b_n\}$ given by J satisfies Re $b_n = 0$ for $n = 1, 2, \ldots$. This is a consequence of the fact that $p(z) = 1 + uz^n$ belongs to $\mathcal{P}_{\mathbf{R}}$ whenever $-1 \leqslant u \leqslant 1$ and $n = 1, 2, \ldots$.

Conversely, each function p of this form (with $m = 1, 2, \ldots$) belongs to supp $\mathcal{P}_{\mathbf{R}}$. To prove this, one begins as in [3, p. 101] by constructing a suitable polynomial F so that $\operatorname{Re} F(x) \geq 0$ for $|x| \leq 1$, and $\operatorname{Re} F(x) = 0$ only when x belongs to the set $\{x_1, x_2, \ldots, x_m, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$. If $H(z) = \frac{1}{2}(F(z) + \overline{F(\bar{z})}) = \sum_{n=0}^{\infty} b_n a^n$, then b_n is real. If we let $c_0 = -b_0$ and $c_n = -b_n/2$ for $n = 1, 2, \ldots,$ then $G(z) = \sum_{n=0}^{\infty} c_n z^n$ defines a suitable functional. Namely

$$J_G\left[\frac{1 - z^2}{(1 - xz)(1 - \bar{x}z)}\right] = J_G\left[\frac{1}{2}\frac{1 + xz}{1 - xz} + \frac{1}{2}\frac{1 + \bar{x}z}{1 - \bar{x}z}\right] = J_G\left[1 + \sum_{n=1}^{\infty} (x^n + \bar{x}^n) z^n\right]$$
$$= c_0 + \sum_{n=1}^{\infty} c_n(x^n + \bar{x}^n) = \operatorname{Re}\left\{c_0 + \sum_{n=1}^{\infty} 2c_n x^n\right\} = \operatorname{Re}\{-F(x)\}.$$

Therefore, $\operatorname{Re} J_G$ is maximized over $\mathfrak{E}\mathcal{P}_{\mathbf{R}}$ only for

$$p(z) = \frac{1 - z^2}{(1 - x_k z)(1 - \bar{x}_k z)} \qquad (k = 1, 2, \ldots, m).$$

This implies that each function of the prescribed form also maximizes Re J_G over $\mathcal{P}_{\mathbf{R}}$ and, so, belongs to supp $\mathcal{P}_{\mathbf{R}}$.

The family T consists of all functions that are analytic in Δ with $f(z)$ is real if and only if z is real $(-1 < z < 1)$ and $f(0) = 0$ and $f'(0) = 1$. The families T and $\mathcal{P}_{\mathbf{R}}$ are in one-to-one correspondence through the relation $f(z) = zp(z)/(1 - z^2)$. It is not difficult to use this correspondence in order to determine supp T from the previous facts about supp $\mathcal{P}_{\mathbf{R}}$. The resulting information is stated in the next theorem.

THEOREM 3. *The set of support points of T consists of all functions of the form*

$$f(z) = \sum_{k=1}^{m} \lambda_k \frac{z}{(1 - x_k z)(1 - \bar{x}_k z)}, \tag{12}$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m} \lambda_k = 1$, $|x_k| = 1$ *and* $\operatorname{Im} x_k \geq 0$ $(m = 1, 2, \ldots)$.

2. Support points where the majorant is univalent or convex.

THEOREM 4. *Let F be analytic in* Δ *and satisfy* $F'(z) \neq 0$ *for* $|z| < 1$, *and let* $\mathfrak{F}$ *be the set of functions that are subordinate to F in* Δ. *If f is a support point of* $\mathfrak{F}$, *then* $f = F \circ \phi$, *where* ϕ *is a finite Blaschke product and* $\phi(0) = 0$.

PROOF. Suppose that $f \in \operatorname{supp} \mathfrak{F}$. Then $f = F \circ \phi$ where $\phi \in \mathfrak{B}_0$, and there is a continuous, linear functional $J\colon \mathcal{Q} \to \mathbf{C}$ so that

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) \colon g \in \mathfrak{F}\} \tag{13}$$

and $\operatorname{Re} J$ is not constant on $\mathfrak{F}$. (13) is the same as

$$\operatorname{Re} J[F \circ \phi] = \max\{\operatorname{Re} J[F \circ \omega] \colon \omega \in \mathfrak{B}_0\}. \tag{14}$$

Define a functional I on $\mathfrak{B}_0$ by $I(\omega) = J[F \circ \omega]$ where $\omega \in \mathfrak{B}_0$. Suppose that $\omega \in \mathfrak{B}_0$, $0 \leq \varepsilon \leq 1$ and note that $(1 - \varepsilon)\phi + \varepsilon\omega \in \mathfrak{B}_0$ as $\mathfrak{B}_0$ is convex. Then, as $\varepsilon \to 0^+$,

$$\begin{aligned} I[(1 - \varepsilon)\phi + \varepsilon\omega] &= I[\phi + (\omega - \phi)\varepsilon] = J[F \circ (\phi + (\omega - \phi)\varepsilon)] \\ &= J[F \circ \phi + F' \circ \phi(\omega - \phi)\varepsilon + o(\varepsilon)] \\ &= J[F \circ \phi] + J[F' \circ \phi(\omega - \phi)]\varepsilon + o(\varepsilon). \end{aligned}$$

Recalling (14) we conclude that $\mathrm{Re}\{J[F' \circ \phi(\omega - \phi)]\varepsilon + o(\varepsilon)\} \leq 0$ whenever $\omega \in \mathfrak{B}_0$ and ε is sufficiently small. Letting $\varepsilon \to 0^+$, we conclude that

$$\mathrm{Re}\, J[(F' \circ \phi)\omega] \leq \mathrm{Re}\, J[(F' \circ \phi)\phi] \tag{15}$$

whenever $\omega \in \mathfrak{B}_0$. A functional L is defined on $\mathfrak{B}_0$ by $L(\omega) = J[(F' \circ \phi)\omega]$, and this functional may be extended to a continuous, linear functional on $\mathcal{Q}$. Because of (15) this implies that $\phi \in \mathrm{supp}\, \mathfrak{B}_0$ if we show that $\mathrm{Re}\, L$ is not constant on $\mathfrak{B}_0$.

Suppose that $\mathrm{Re}\, L$ is constant on $\mathfrak{B}_0$. Since $xz^n \in \mathfrak{B}_0$ for $|x| = 1$ and $n = 1, 2, \ldots$, we deduce that $L = 0$ on $\mathcal{Q}_0$. But since $F'(z) \neq 0$ for $|z| < 1$, we know that $z^n/(F' \circ \phi) \in \mathcal{Q}_0$ and, so, $L[z^n/(F' \circ \phi)] = 0$. Thus, $J(z^n) = 0$ for $n = 1, 2, \ldots$, which implies that J and, hence, $\mathrm{Re}\, J$ is zero on $\mathcal{Q}_0$. If $f \in \mathfrak{F}$, then $f = F(0) + g$, where $g \in \mathcal{Q}_0$ and so $\mathrm{Re}\, J$ is constant on $\mathfrak{F}$. This contradiction completes the argument that $\phi \in \mathrm{supp}\, \mathfrak{B}_0$. Theorem 2 yields the conclusion that ϕ is a finite Blaschke product.

REMARK. Theorem 4 holds, in particular, if F is analytic and univalent in Δ.

THEOREM 5. *Let F be analytic, univalent and convex in Δ and let $\mathfrak{F}$ denote the set of functions subordinate to F in Δ. If ϕ is a finite Blaschke product with $\phi(0) = 0$, then $f = F \circ \phi$ is a support point of $\mathfrak{F}$.*

PROOF. Suppose that $\phi \in \mathfrak{B}_0$ and $f = F \circ \phi$, and let

$$\phi(z) = \sum_{n=1}^{\infty} c_n z^n, \quad F(z) = \sum_{n=0}^{\infty} A_n z^n \quad \text{and} \quad f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then $a_0 = A_0$, $a_1 = A_1 c_1$, $a_2 = A_1 c_2 + A_2 c_1^2$ and, in general, for $k \geq 1$, $a_k = A_1 c_k + \Phi_k(c_1, c_2, \ldots, c_{k-1})$, where Φ_k is a polynomial with coefficients depending on $A_2, A_3, \ldots, A_k$. Let C_n denote the set of points $(d_0, d_1, \ldots, d_{n-1})$ so that there is a function ω in $\mathfrak{B}$ for which $\omega(z) = d_0 + d_1 z + \cdots + d_{n-1} z^{n-1} + \cdots$. Also let D_n denote the corresponding set $(a_1, a_2, \ldots, a_n)$ so that $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ for some f in $\mathfrak{F}$. The mapping $(c_1, c_2, \ldots, c_n) \to (a_1, a_2, \ldots, a_n)$ of $\mathbf{C}^n$ to $\mathbf{C}^n$, given by $a_k = A_1 c_k + \Phi_k(c_1, c_2, \ldots, c_{k-1})$ for $k = 1, 2, \ldots, n$ when restricted to C_n, defines a homeomorphism from C_n onto D_n. We note that $\phi \in \mathfrak{B}_0$ if and only if $\phi(z)/z \in \mathfrak{B}$ and, since $A_1 \neq 0$, we may solve the system

$$a_1 = A_1 c_1, \quad a_2 = A_1 c_2 + \Phi_1(c_1), \ldots, \quad a_n = A_1 c_n + \Phi_n(c_1, c_2, \ldots, c_{n-1})$$

recursively for $c_1, c_2, \ldots, c_n$ in terms of $a_1, a_2, \ldots, a_n$.

Because F is convex, the family $\mathfrak{F}$ is convex and thus D_n is a convex subset of $\mathbf{C}^n$. Since C_n has a nonvoid interior, D_n also has a nonvoid interior and so D_n is a convex body in $\mathbf{C}^n$. If $(a_1', a_2', \ldots, a_n') \in \partial D_n$ then there is a support plane to D_n through $(a_1', a_2', \ldots, a_n')$. Thus, there are complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, so that

$$\mathrm{Re} \sum_{k=1}^{n} \alpha_k a_k \leq \mathrm{Re} \sum_{k=1}^{n} \alpha_k a_k' \tag{16}$$

whenever $(a_1, a_2, \ldots, a_n) \in D_n$. If we let $J(f) = \sum_{k=1}^{n} \alpha_k a_k$ whenever $f \in \mathcal{Q}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then J is a continuous, linear functional on $\mathcal{Q}$, and (16) may be written

$$\mathrm{Re}\, J(f_0) = \max\{\mathrm{Re}\, J(f) : f \in \mathfrak{F}\}, \tag{17}$$

where f_0 is any function in $\mathfrak{F}$ with a power series beginning $f_0(z) = a_0 + \Sigma_{k=1}^{n} a'_k z^k + \cdots$. Because of the homeomorphism between C_n and D_n, such a function f_0 must be associated with a function ϕ_0 so that if $\phi_0(z) = \Sigma_{k=1}^{n} c'_k z^k + \cdots$, then $(c'_1, c'_2, \ldots, c'_n) \in \partial C_n$. This implies that ϕ_0 is a finite Blaschke product with $\phi_0(z)/z$ having the form (5) with $m = n$ [**4**, p. 79].

Since the functions $\phi_0(z)/z$ of that form are in one-to-one correspondence with ∂C_n, we see that every such function produces a function f_0 that solves an equation (17) for a continuous, linear functional J. Since D_n has a nonvoid interior, Re J is not constant on $\mathfrak{F}$. Hence $f_0 \in \operatorname{supp} \mathfrak{F}$.

COROLLARY 1. *Let the function F be analytic, univalent and convex in* Δ *and let* $\mathfrak{F} = \{f\colon f \prec F \text{ in } \Delta\}$. *Then,* supp $\mathfrak{F}$ *consists of all functions* $F \circ \phi$ *where* ϕ *is a finite Blaschke product and* $\phi(0) = 0$.

PROOF. This is an immediate consequence of Theorems 4 and 5.

THEOREM 6. *Let the function F be analytic, univalent and convex in* Δ *and assume that* $F(\Delta)$ *is not a half-plane. Let* $\mathfrak{F} = \{f\colon f \prec F \text{ in } \Delta\}$, *and let J be a continuous, linear functional on* $\mathfrak{A}$ *so that* Re J *is not constant on* $\mathfrak{F}$. *Then there is a unique function f in* $\mathfrak{F}$ *so that*

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g)\colon g \in \mathfrak{F}\}. \tag{18}$$

PROOF. Suppose that f and f_1 belong to $\mathfrak{F}$ and satisfy (18). If $0 \leqslant t \leqslant 1$, then $h = tf + (1-t)f_1 \in \mathfrak{F}$ since F is convex. Clearly, h also satisfies (18), and, by Theorem 4, we conclude that $h = F \circ \psi$, where ψ is a finite Blaschke product and $\psi(0) = 0$. Since $F(\Delta)$ is not a half-plane, $F \in H^p$ for some $p > 1$ [**7**, p. 467] and, therefore [**7**, p. 465], h is an extreme point of $\mathfrak{F}$ (ψ being a finite Blaschke product is, in particular, an inner function). Hence, the relation $h = tf + (1-t)f_1$ $(0 < t < 1)$ implies that $f = f_1$.

REMARK. Theorem 6 implies that $\operatorname{supp} \mathfrak{F} \subset \mathfrak{E}\mathfrak{F}$. These two sets are usually distinct. An example of a situation where Corollary 1 and Theorem 6 are applicable is given by the family of analytic functions having a range in a prescribed angular region with opening less than Π and having fixed values at 0. The majorant $F(z) = ((1 + cz)/(1 - z))^p$, where $|c| \leqslant 1$, $c \neq -1$ and $0 < p < 1$, defines such a family. Also, the extreme points correspond to inner functions φ [**7**, p. 465 and **1**], and the support points correspond to the special inner functions given by finite Blaschke products. In §3 we consider the same majorant in the case $p > 1$. Theorem 6 may be generalized to Fréchet differentiable functionals J if the derivative of J does not have a constant real part on $\mathfrak{F}$.

The final result in this section demonstrates that F need not have restricted growth, such as $|F(z)|(1 - |z|) = O(1)$ as $|z| \to 1$, in order that $\operatorname{supp} \mathfrak{F}$ be a fairly diverse set. This contrasts with Theorem 8 in §3 where $\operatorname{supp} \mathfrak{F} = \{F(xz)\colon |x| = 1\}$. We state the result with $p > 1$ since, according to Theorem 5, the example $F(z) = 1/(1 - z)^p$ with $0 < p < 1$ would yield the result (even for all n).

THEOREM 7. *Suppose that $p > 1$ and n is a positive integer. There is a function F such that $(1 - z)^p F(z)$ is analytic in $\bar{\Delta}$ and does not vanish at $z = 1$. Moreover, if $\mathfrak{F} = \{f: f \prec F \text{ in } \Delta\}$, then each function $f = F \circ \phi$, where ϕ is a finite Blaschke product of degree at most n and $\phi(0) = 0$, belongs to $\mathfrak{E}\mathfrak{H}(\mathfrak{F})$ and to* supp $\mathfrak{F}$.

PROOF. Let F, G and H belong to $\mathfrak{A}$ so that $H = FG$, and let

$$F(z) = \sum_{k=0}^{\infty} A_k z^k, \quad G(z) = \sum_{k=0}^{\infty} B_k z^k \quad \text{and} \quad H(z) = \sum_{k=0}^{\infty} C_k z^k.$$

Then $C_k = \Sigma_{j=0}^{k} A_j B_{k-j}$. If $B_0 \neq 0$ and $C_k = \Sigma_{j=0}^{k-1} A_j B_{k-j}$ for $k = 2, 3, \ldots, n$, this implies that $A_k = 0$ for $k = 2, 3, \ldots, n$. In other words, if $G(z) \neq 0$ for $|z| < 1$ and if $C_k = \Sigma_{j=0}^{k-1} A_j B_{k-j}$ for $k = 2, 3, \ldots, n$, then the function $F = H/G$ is analytic in Δ and $A_k = 0$ for $k = 2, 3, \ldots, n$. By letting $H(z) = \Sigma_{k=0}^{n+1} C_k z^k$, we are also assured that $H(1) \neq 0$ simply by choosing $C_{n+1} \neq -\Sigma_{k=0}^{n} C_k$.

In particular, if $G(z) = (1 - z)^p$ then there is a polynomial H of degree at most $n + 1$ so that $H(1) \neq 0$ and

$$F(z) = H(z)/G(z) = A_0 + A_1 z + A_{n+1} z^{n+1} + \cdots.$$

Now, suppose that $\phi(z) = \Sigma_{k=1}^{\infty} d_k z^k$ belongs to $\mathfrak{B}_0$ and $f(z) = F(\phi(z)) = \Sigma_{k=0}^{\infty} a_k z^k$. Then the coefficient relations described in Theorem 5 imply $a_0 = A_0$ and

$$a_1 = A_1 d_1, \quad a_2 = A_1 d_2, \ldots, \quad a_n = A_1 d_n. \tag{19}$$

The equations $C_0 = A_0 B_0$ and $C_1 = A_0 B_1 + A_1 B_0$ become $C_0 = A_0$ and $C_1 = -pA_0 + A_1$ and, so, $A_1 = C_1 + pC_0$. By choosing $C_1 \neq -pC_0$ we get $A_1 \neq 0$. The mapping given by (19) shows that the coefficient region $\{(a_1, a_2, \ldots, a_n)\}$ is a convex body in $\mathbf{C}^n$ and through each of its boundary points there is a support plane which intersects the region only at that point. This is a consequence of the same properties of the coefficient regions of $\mathfrak{B}$ [**4**, p.80]. Therefore, if ϕ_0 is a finite Blaschke product of degree at most n and $\phi_0(0) = 0$, then $f_0 = F \circ \phi_0$ is the unique function in $\mathfrak{F}$ so that

$$\operatorname{Re} J(f_0) = \max\{\operatorname{Re} J(f): f \in \mathfrak{F}\} \tag{20}$$

for a functional J determined by a support plane. Re J is nonconstant on $\mathfrak{F}$ as $\{(a_1, a_2, \ldots, a_n)\}$ has a nonvoid interior. This implies that $f_0 \in$ supp $\mathfrak{F}$ and also $f \in \mathfrak{E}\mathfrak{H}\mathfrak{F}$.

3. The case $F(z) = ((1 + cz)/(1 - z))^p$, $p > 1$, **and related families.** We recall the Herglotz formulas which asserts that $p \in \mathfrak{P}$ if and only if there exists a probability measure μ on $X = \{x: |x| = 1\}$ such that

$$p(z) = \int_X \frac{1 + xz}{1 - xz}\, d\mu(x). \tag{21}$$

A consequence of this formula is the relation $\mathfrak{E}\mathfrak{P} = \{(1 + xz)/(1 - xz): |x| = 1\}$. We recall that Theorem 1 implies that $\mathfrak{E}\mathfrak{P} \subseteq$ supp $\mathfrak{P}$.

Suppose that $p > 1$, $|c| \leq 1$, $c \neq -1$ and $F(z) = ((1 + cz)/(1 - z))^p$, and let $\mathfrak{F} = \{f: f \prec F \text{ in } \Delta\}$. It is known [**2**] that $\mathfrak{E}\mathfrak{H}\mathfrak{F} = \{((1 + cxz)/(1 - xz))^p: |x| = 1\}$ and, hence, for each function f in $\mathfrak{F}$ there is a probability measure μ on X such that $f(x) = \int_X ((1 + cxz)/(1 - xz))^p\, d\mu(x)$. The main theorem of this section is the assertion supp $\mathcal{F} = \mathfrak{E}\mathfrak{H}\mathcal{F}$, a result which contrasts sharply with the case $p = 1$ as seen in Theorem 1.

In the process of proving this theorem, we develop a technique, concerning the measures μ, for deciding when certain functions do not belong to a given family. The first lemma was obtained by Louis Brickman a number of years ago, and we present his argument.

LEMMA 2. *Suppose that $p \in \mathcal{P}$ is given by* (21). *Let A be any open arc on X and let $\bar{A}$ denote the union of A with its two endpoints. Then*

$$\frac{1}{2}\left[\mu(A) + \mu(\bar{A})\right] = \lim_{r\to 1^-}\int_A \operatorname{Re} p(re^{i\theta})\,\frac{d\theta}{2\pi}. \tag{22}$$

PROOF. Let χ_A be the characteristic function of A, and let x_1 and x_2 denote the endpoints of A. If $0 < r < 1$ then

$$\int_0^{2\pi}\chi_A(e^{i\theta})\operatorname{Re} p(re^{i\theta})\,\frac{d\theta}{2\pi} = \int_0^{2\pi}\int_X \chi_A(e^{i\theta})\operatorname{Re}\left(\frac{1+xre^{i\theta}}{1-xre^{i\theta}}\right)d\mu(x)\,\frac{d\theta}{2\pi}.$$

By Fubini's theorem this equality implies that

$$\int_A \operatorname{Re} p(re^{i\theta})\,\frac{d\theta}{2\pi} = \int_X\left\{\int_0^{2\pi}\chi_A(e^{i\theta})\operatorname{Re}\left(\frac{1+xre^{i\theta}}{1-xre^{i\theta}}\right)\frac{d\theta}{2\pi}\right\}d\mu(x).$$

If the harmonic function u is defined in Δ by the Poisson formula with boundary function v belonging to $L^1[0,2\pi]$, then $\lim_{r\to 1^-} u(re^{i\theta}) = \frac{1}{2}[v(\theta^+) + v(\theta^-)]$ at each point θ where v has right- and left-hand limits. Therefore,

$$\lim_{r\to 1^-}\int_0^{2\pi}\chi_A(e^{i\theta})\operatorname{Re}\left\{\frac{1+e^{i\phi}re^{i\theta}}{1-e^{i\phi}re^{i\theta}}\right\}\frac{d\theta}{2\pi} = \begin{cases} 0 & \text{if } e^{i\phi}\notin \bar{A},\\ 1 & \text{if } e^{i\phi}\in A,\\ \frac{1}{2} & \text{if } e^{i\phi} = x_1 \text{ or } x_2.\end{cases}$$

This implies that

$$\begin{aligned}\lim_{r\to 1^-}\int_A \operatorname{Re} p(re^{i\theta})\,\frac{d\theta}{2\pi} &= \int_X \lim_{r\to 1^-}\left\{\int_0^{2\pi}\chi_A(e^{i\theta})\operatorname{Re}\left(\frac{1+xre^{i\theta}}{1-xre^{i\theta}}\right)\frac{d\theta}{2\pi}\right\}d\mu(x)\\ &= \int_A 1\,d\mu(x) + \int_{\{x_1\}}\frac{1}{2}\,d\mu(x) + \int_{\{x_2\}}\frac{1}{2}\,d\mu(x)\\ &= \mu(A) + \frac{1}{2}\mu[\{x_1\}] + \frac{1}{2}\mu[\{x_2\}] = \frac{1}{2}\left[\mu(A) + \mu(\bar{A})\right],\end{aligned}$$

where the first equality is valid by Lebesgue's bounded convergence theorem.

LEMMA 3. *Suppose that $p \in \mathcal{P}$ is given by* (21). *If $|x_0| = 1$ and p has a pole at x_0 with residue R, then $R/x_0 < 0$ and $\mu[\{x_0\}] = -\frac{1}{2}(R/x_0)$.*

PROOF. Let $\{A_n\}$ be a monotone decreasing sequence of open arcs on X so that $\bigcap_{n=1}^{\infty} A_n = \{x_0\}$. Then, as $n\to\infty$,

$$\frac{1}{2}\left[\mu(A_n) + \mu(\bar{A}_n)\right] \to \frac{1}{2}\left(\mu[\{x_0\}] + \mu[\{x_0\}]\right) = \mu[\{x_0\}].$$

Lemma 2 implies that

$$\mu[\{x_0\}] = \lim_{n\to\infty}\left(\lim_{r\to 1^-}\int_{A_n} \operatorname{Re} p(re^{i\theta})\,\frac{d\theta}{2\pi}\right). \tag{23}$$

If (23) is applied to the particular function $p(z) = (1 + \bar{x}_0 z)/(1 - \bar{x}_0 z)$, then, as $\mu[\{x_0\}] = 1$, we conclude that

$$\lim_{n\to\infty}\left(\lim_{r\to 1^-}\int_{A_n} \operatorname{Re}\left[\frac{1+\bar{x}_0 re^{i\theta}}{1-\bar{x}_0 re^{i\theta}}\right]\frac{d\theta}{2\pi}\right) = 1. \tag{24}$$

If p has a pole at x_0 with residue R, then $p(z) = R/(z - x_0) + q(z)$, where q is analytic at x_0. This may be written

$$p(z) = -\frac{R}{2x_0}\,\frac{1+\bar{x}_0 z}{1-\bar{x}_0 z} + s(z) \tag{25}$$

where s is analytic at x_0. Since $w = (1 + \bar{x}_0 z)/(1 - \bar{x}_0 z)$ maps Δ onto $\{w\colon \operatorname{Re} w > 0\}$ and s is bounded near x_0, the condition $\operatorname{Re} p(z) > 0$ for $|z| < 1$ implies that $-R/x_0 > 0$. Otherwise, there are points z in Δ near x_0 for which $\operatorname{Re} p(z)$ may tend to $-\infty$.

(23) and (24) may be applied to (25) and this yields

$$\begin{aligned}\mu[\{x_0\}] &= \lim_{n\to\infty}\left(\lim_{r\to 1^-}\int_{A_n}\operatorname{Re}\left[-\frac{R}{2x_0}\left(\frac{1+\bar{x}_0 re^{i\theta}}{1-\bar{x}_0 re^{i\theta}}\right) + s(re^{i\theta})\right]\frac{d\theta}{2\pi}\right)\\ &= -\frac{R}{2x_0} + \lim_{n\to\infty}\left(\lim_{r\to 1^-}\int_{A_n}\operatorname{Re} s(re^{i\theta})\,\frac{d\theta}{2\pi}\right).\end{aligned}$$

The last limit is zero because $\operatorname{Re} s$ is continuous at $z = x_0$. Thus, $\mu[\{x_0\}] = -R/2x_0$.

LEMMA 4. *Suppose that* $p > 1$, $|c| \leq 1$, $c \neq -1$ *and* $F(z) = ((1 + cz)/(1 - z))^p$. *Let*

$$f(z) = \sum_{k=1}^{n} \lambda_k \left(\frac{1 + cx_k z}{1 - x_k z}\right)^p$$

where $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k = 1$, $|x_k| = 1$ *and* $n \geq 2$. *Then* f *is not subordinate to* F *in* Δ.

PROOF. On the contrary, assume that f is subordinate to F in Δ. This implies that $f(z) \neq 0$ for $|z| < 1$ since $F(z) \neq 0$ for $|z| < 1$. Thus, $g = f^{1/p}$ is analytic in Δ and $g \prec F^{1/p} = G$, where $G(z) = (1 + cz)/(1 - z)$. Since

$$G(z) = \frac{1+c}{2}\,\frac{1+z}{1-z} + \frac{1-c}{2},$$

we may write

$$g = \frac{1+c}{2}q + \frac{1-c}{2}, \quad \text{where } q(z) = \int_X \frac{1+xz}{1-xz}\,d\mu(x)$$

and μ is a probability measure on X. Since $f(z) = \lambda_k((1 + cx_k z)/(1 - x_k z))^p + f_k(z)$, where f_k is analytic at $\bar{x}_k$, it follows that $g(z) = \lambda_k^{1/p}(1 + cx_k z)/(1 - x_k z) + g_k(z)$, where g_k is defined and continuous in $N = \{z\colon |z - \bar{x}_k| < \varepsilon\} \cap \bar{\Delta}$ for sufficiently small ε (in fact, $g_k(\bar{x}_k) = 0$ as $p > 1$). Thus, $q(z) = \lambda_k^{1/p}(1 + x_k z)/(1 - x_k z) + q_k(z)$, and q_k is continuous in N. The arguments given in the proof of Lemma 3 imply that $\mu[\{\bar{x}_k\}] = \lambda_k^{1/p}$. Therefore,

$$\mu(X) \geqslant \sum_{k=1}^{n} \mu[\{\bar{x}_k\}] = \sum_{k=1}^{n} \lambda_k^{1/p} > \sum_{k=1}^{n} \lambda_k = 1.$$

This uses the fact that if $0 < \lambda < 1$ and $p > 1$, then $\lambda^{1/p} > \lambda$. The inequality $\mu(X) > 1$ is impossible since $\mu(X) = 1$.

THEOREM 8. *Suppose that $p > 1$, $|c| \leqslant 1$, $c \neq -1$, $F(z) = ((1 + cz)/(1 - z))^p$ and $\mathfrak{F}$ denotes the set of functions that are subordinate to F in Δ. Then* $\operatorname{supp} \mathfrak{F} = \mathfrak{E}\mathfrak{H}\mathfrak{F} = \{F(xz) \colon |x| = 1\}$.

PROOF. The inclusion $\mathfrak{E}\mathfrak{H}\mathfrak{F} \subset \operatorname{supp} \mathfrak{F}$ follows from the general fact that the functions $F(xz)$ ($|x| = 1$) are support points for the class of functions subordinate to a nonconstant function F [7, p. 463] and the relation $\mathfrak{E}\mathfrak{H}\mathfrak{F} = \{((1 + cxz)/(1 - xz))^p \colon |x| = 1\}$ mentioned earlier.

Suppose that J is a continuous, linear functional on $\mathfrak{A}$ so that $\operatorname{Re} J$ is nonconstant on $\mathfrak{F}$ and J is given by the sequence $\{b_n\}$. Let

$$F(z) = 1 + \sum_{n=1}^{\infty} A_n z^n \quad \text{and} \quad F(xz) = 1 + \sum_{n=1}^{\infty} A_n x^n z^n.$$

Then

$$J[F(xz)] = b_0 + \sum_{n=1}^{\infty} A_n b_n x^n = G(x)$$

defines a function G analytic in $\bar{\Delta}$, since $\overline{\lim}_{n\to\infty} \sqrt[n]{|A_n b_n|} \leqslant \overline{\lim}_{n\to\infty} \sqrt[n]{|b_n|} < 1$. G is nonconstant, otherwise $\operatorname{Re} J$ is constant on $\mathfrak{H}\mathfrak{F}$. Consequently, if we let $M = \max\{\operatorname{Re} J(f) \colon f \in \mathfrak{F}\}$, then $\operatorname{Re} G(x) = M$ has only a finite number of solutions with $|x| = 1$. By familiar arguments [**3**, p.100] we conclude that the solution set over $\mathfrak{H}\mathfrak{F}$ is the convex hull of a finite number of extreme points $F(xz)$, that is, it consists of the functions given in Lemma 4 where $0 \leqslant \lambda_k \leqslant 1$. If at least two λ_k's are nonzero, by applying Lemma 4 we get a contradiction. Thus, the only functions in $\mathfrak{H}\mathfrak{F}$ which belong to the solution set and also to $\mathfrak{F}$ are the functions $((1 + cx_k z)/(1 - x_k z))^p$, certain extreme points of $\mathfrak{H}\mathfrak{F}$.

REMARKS. 1. If $\{((1 + cx_k z)/(1 - x_k z))^p \colon k = 1, 2, \ldots, n\}$ is a finite collection from $\mathfrak{E}\mathfrak{H}\mathfrak{F}$, then by familiar arguments [**3**, p. 101] it is easy to construct a nontrivial continuous, linear functional J so that the solution set over $\mathfrak{F}$ consists of the given collection.

2. An alternative proof of Theorem 8 which does not use the technique of Lemma 3 may be obtained from Theorem 3 by the kind of argument used later in the proof of Theorem 10.

3. Let $R(p, \alpha)$ denote the set of functions in $\mathfrak{A}$ which satisfy $f(0) = 0$, $f'(0) = 1$ and $\operatorname{Re} \sqrt[p]{f(z)/z} > \alpha$ for $|z| < 1$ ($0 \leqslant \alpha < 1$). As mentioned in [**7**, p. 461] for the case $p = 2$, and which in fact is true for any $p > 1$,

$$\mathfrak{E}\mathfrak{H}R(p, \alpha) = \left\{((1 + (1 - 2\alpha)xz)/(1 - xz))^p \colon |x| = 1\right\}.$$

Applying the technique for proving Theorem 8, we conclude that $\operatorname{supp} R(p, \alpha) = \mathfrak{E}\mathfrak{H}R(p, \alpha)$.

THEOREM 9. *Suppose that* $p > 1$, $F(z) = ((1+z)/(1-z))^p$, *and let* $\mathfrak{F}$ *denote the set of functions that are subordinate to* F *in* Δ *and satisfy* $f(z)$ *is real when* z *is real* $(-1 < z < 1)$. *Then*

$$\operatorname{supp}\mathfrak{F} = \mathfrak{E}\mathfrak{H}\mathfrak{F} = \left\{\left[\frac{1-z^2}{(1-xz)(1-\bar{x}z)}\right]^p : |x| = 1, \operatorname{Im} x \geq 0\right\}.$$

PROOF. It follows easily from results proved in [**6**, p. 168] about $\mathfrak{H}\mathfrak{F}$ that $\mathfrak{E}\mathfrak{H}\mathfrak{F}$ is the set described in the theorem. Let J be a continuous, linear functional on $\mathfrak{A}$ with $\operatorname{Re} J$ nonconstant on $\mathfrak{F}$, and let

$$F(x) = J\left[\{1 - z^2/(1-xz)(1-z/x)\}^p\right]$$

where $p > 1$ and $|x| = 1$. There exists [**11**, p. 36] a finite complex Borel measure μ with compact support C contained in Δ such that

$$F(x) = \int_C \left[1 - z^2/(1-x\zeta)(1-\zeta/x)\right]^p d\mu(\zeta).$$

Hence F can be extended to be analytic in an annulus containing $\partial\Delta$. Also, let $G(x) = \frac{1}{2}[F(x) + \overline{F(1/\bar{x})}]$; then G is analytic on $\partial\Delta$ and $G(x) = \operatorname{Re} F(x)$ whenever $|x| = 1$.

Let $M = \max\{\operatorname{Re} J(f): f \in \mathfrak{F}\}$. We claim that there are only a finite number of solutions to $\operatorname{Re} F(x) = M$ with $|x| = 1$. Otherwise, since $G(x) = \operatorname{Re} F(x)$ for $|x| = 1$, the identity theorem implies that $G = M$ and so $\operatorname{Re} F(x) = M$ for $|x| = 1$. Since $\mathfrak{E}\mathfrak{H}\mathfrak{F} = \{[(1-z^2)/(1-xz)(1-\bar{x}z)]^p\}$, this implies that $\operatorname{Re} J(f) = M$ for all f in $\mathfrak{H}\mathfrak{F}$, and, in particular, $\operatorname{Re} J$ is constant on $\mathfrak{F}$.

Hence the solution set over $\mathfrak{H}\mathfrak{F}$ is the convex hull of a finite number of the extreme points. Using the argument given in Lemma 4, it is not difficult to show that if

$$f(z) = \sum_{k=1}^n \lambda_k \left[\frac{1-z^2}{(1-x_k z)(1-\bar{x}_k z)}\right]^p \qquad \left(\lambda_k \geq 0,\ \sum_{k=1}^n \lambda_k = 1, |x_k| = 1\right),$$

then $f \in \mathfrak{F}$ if and only if $\lambda_k = 1$ for some k. This proves that $\operatorname{supp}\mathfrak{F} \subset \mathfrak{E}\mathfrak{H}\mathfrak{F}$.

Now we show that $\mathfrak{E}\mathfrak{H}\mathfrak{F} \subset \operatorname{supp}\mathfrak{F}$. If $f(z) = [(1-z^2)/(1-xz)(1-\bar{x}z)]^p = 1 + a_1 z + a_2 z^2 + \cdots$, then

$$a_1 = p(x + \bar{x}) \quad \text{and} \quad a_2 = p(x^2 + \bar{x}^2) + p(p-1)(x+\bar{x})^2/2.$$

Letting $x = e^{i\theta}$, $0 \leq \theta \leq \pi$, and $u = \cos\theta$, we may write $a_1 = 2pu$ and $a_2 = 2p(p+1)u^2 - 2p$. If $b = -1/2(p+1)$ and a is real, then

$$A(u) = aa_1 + ba_2 = 2p\left[-\tfrac{1}{2}u^2 + au - b\right]$$

achieves its maximum value at $u = a$. By letting a be any number in $[-1, 1]$, we see that each function in $\mathfrak{E}\mathfrak{H}\mathfrak{F}$ is the unique solution to an extremal problem $\max\{\operatorname{Re} J(f): f \in \mathfrak{E}\mathfrak{H}\mathfrak{F}\}$ where $J(f) = af'(0) + 2bf''(0)$. This implies that each extreme point is a support point.

THEOREM 10. *Suppose that* $|c| \leq 1$, $c \neq -1$, $F(z) = \exp((1+cz)/(1-z))$, *and* $\mathfrak{F}$ *denotes the set of functions that are subordinate to* F *in* Δ. *Then* $\operatorname{supp}\mathfrak{F} = \mathfrak{E}\mathfrak{H}\mathfrak{F} = \{F(xz): |x| = 1\}$.

PROOF. The inclusion $\mathfrak{E}\mathfrak{H}\mathfrak{F} \subset \operatorname{supp}\mathfrak{F}$ follows from the general fact that $\{F(xz): |x|=1\}$ are support points whenever the majorant F is nonconstant [7, p. 463] and the result that $\mathfrak{E}\mathfrak{H}\mathfrak{F} = \{F(xz): |x|=1\}$ [5] for the given majorant.

Suppose that J is a continuous, linear functional on $\mathfrak{A}$ with $\operatorname{Re} J$ nonconstant on $\mathfrak{F}$, and let J be given by the sequence $\{b_n\}$. If $F(z) = \sum_{n=0}^{\infty} A_n z^n$, then $J(F(xz)) = \sum_{n=0}^{\infty} A_n b_n x^n = G(x)$ defines a function G analytic in $\bar{\Delta}$ which is clearly nonconstant. Therefore, if we let $M = \max\{\operatorname{Re} J(f): f \in \mathfrak{F}\}$, then there are only a finite number of solutions to $\operatorname{Re} G(x) = M$ with $|x|=1$. By a familiar argument [3, p. 100], the only functions in the solution set over $\mathfrak{H}\mathfrak{F}$ have the form

$$f(z) = \sum_{k=1}^{n} \lambda_k \exp\left(\frac{1+cx_k z}{1-x_k z}\right) \quad \text{where } \lambda_k \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1 \text{ and } |x_k| = 1.$$

We shall show that if f has such a form and is in $\operatorname{supp}\mathfrak{F}$, then $\lambda_k = 1$ for some k.

Since $F'(z) \neq 0$ for $|z|<1$, we may apply Theorem 4 to conclude that $f(z) = \exp((1+c\phi(z))/(1-\phi(z)))$ and ϕ is a finite Blaschke product with $\phi(0) = 0$. We have

$$\sum_{k=1}^{n} \lambda_k \exp\left(\frac{1+cx_k z}{1-x_k z}\right) = \exp\left(\frac{1+c\phi(z)}{1-\phi(z)}\right),$$

and, since the function on the right-hand side of the equality has no zeros in Δ, we may write

$$\log\left\{\sum_{k=1}^{n} \lambda_k \exp\left(\frac{1+cx_k z}{1-x_k z}\right)\right\} = \frac{1+c\phi(z)}{1-\phi(z)}.$$

It is known [4, p. 83] that there exist numbers t_k, x'_k $(k = 1, 2, \ldots, m)$ so that $t_k \geq 0$, $\sum_{k=1}^{m} t_k = 1$, $|x'_k| = 1$ and

$$\frac{1+c\phi(z)}{1-\phi(z)} = \sum_{k=1}^{m} t_k \frac{1+cx'_k z}{1-x'_k z}.$$

It is now clear that the collections $\{x_1, x_2, \ldots, x_n\}$ and $\{x'_1, x'_2, \ldots, x'_m\}$ coincide so that by relabeling, if necessary, we have

$$\log\left\{\sum_{k=1}^{n} \lambda_k \exp\left(\frac{1+cx_k z}{1-x_k z}\right)\right\} = \sum_{k=1}^{n} t_k \frac{1+cx_k z}{1-x_k z}.$$

Suppose that $\lambda_j > 0$. There exist functions g and h analytic at $\bar{x}_j$ so that

$$\log\left\{\lambda_j \exp\left(\frac{1+cx_j z}{1-x_j z}\right) + g(z)\right\} = -\frac{t_j \bar{x}_j (1+c)}{z-\bar{x}_j} + h(z).$$

By differentiating both sides and rewriting, we obtain

$$(26) \qquad \frac{(1+c)\bar{x}_j + (z-\bar{x}_j)^2 g'(z)/\lambda_j \exp\big((1+cx_j z)/(1-x_j z)\big)}{1 + g(z)/\lambda_j \exp\big((1+cx_j z)/(1-x_j z)\big)}$$
$$= t_j \bar{x}_j (1+c) + (z-\bar{x}_j)^2 h'(z).$$

The ray $\{w: w \geq 1\}$ is contained in $k(\Delta)$, where $k(z) = (1+cx_j z)/(1-x_j z)$. We may choose a sequence $\{z_n\}$ so that $z_n \to \bar{x}_j$, $k(z_n)$ is real and $k(z_n) \to +\infty$. Thus

$\exp[k(z_n)] \to +\infty$, and by letting $z = z_n$ in (26) and then taking the limits, we obtain $(1+c)\bar{x}_j = (1+c)\bar{x}_j t_j$. Therefore, $t_j = 1$ and then $\lambda_j = 1$ and $t_k = \lambda_k = 0$ for $k \neq j$.

REMARK. If $c = 1 - 2\alpha$ and $0 < \alpha < 1$, then Theorem 8, where $1 < p < 3$, and Theorem 10 can be proven by elementary geometric arguments. These arguments, more generally, are applicable to the situation where the majorant has the property that $\mathbf{C} \setminus F(\Delta)$ is convex and $\partial F(\Delta)$ contains no straight line segments. These conditions and $\mathfrak{E}\mathfrak{H}\mathfrak{F} = \{F(xz): |x| = 1\}$ imply that $\operatorname{supp} \mathfrak{F} = \mathfrak{E}\mathfrak{H}\mathfrak{F}$.

4. Support points of K^* and St*. We recall that

$$\mathfrak{E}\mathfrak{H}\mathrm{St}^* = \left\{\frac{xz}{(1-yz)^2} : |x| = |y| = 1\right\} \quad \text{and} \quad \mathfrak{E}\mathfrak{H}K^* = \left\{\frac{xz}{1-yz} : |x| = |y| = 1\right\}.$$

We also shall need the result that if $f \prec g$ in Δ and $g \in \mathrm{St}$, then

$$(\mathfrak{L}f)(z) = \int_0^z \frac{f(\zeta)}{\zeta}\,d\zeta \prec \int_0^z \frac{g(\zeta)}{\zeta}\,d\zeta = (\mathfrak{L}g)(z) \qquad [\mathbf{12}, \text{p. } 777].$$

Since $\mathfrak{L}g \in K$ whenever $g \in \mathrm{St}$, this says that the linear homeomorphism $\mathfrak{L}: \mathfrak{A}_0 \to \mathfrak{A}_0$ maps St* into K^*. It is easy to verify that $\mathfrak{L}(\mathfrak{E}\mathfrak{H}\mathrm{St}^*) = \mathfrak{E}\mathfrak{H}K^*$ and, consequently, $\mathfrak{L}(\mathfrak{H}\mathrm{St}^*) = \mathfrak{H}K^*$.

LEMMA 5. *Let J_F be a continuous, linear functional on $\mathfrak{A}$ given by $F(z) = \sum_{n=0}^{\infty} b_n z^n$, where $\overline{\lim}_{n\to\infty} \sqrt[n]{|b_n|} < 1$, and let $G(z) = zF'(z)$. Then G defines J_G, a continuous, linear functional on $\mathfrak{A}$, and*

$$\max\{\operatorname{Re} J_F(f): f \in \mathrm{St}^*\} = \max\{\operatorname{Re} J_G(g): g \in K^*\}. \tag{27}$$

Also, if $\operatorname{Re} J_F(f_0) = \max\{\operatorname{Re} J_F(f): f \in \mathrm{St}^*\}$, *then*

$$\operatorname{Re} J_G(\mathfrak{L}f_0) = \max\{\operatorname{Re} J_G(g): g \in K^*\}.$$

If $f_0 \in \operatorname{supp} \mathrm{St}^*$ *then* $\mathfrak{L}(f_0) \in \operatorname{supp} K^*$.

PROOF.

$$\begin{aligned} \max\{\operatorname{Re} J_F(f): f \in \mathrm{St}^*\} &= \max\{\operatorname{Re} J_F(f): f \in \mathfrak{H}\mathrm{St}^*\} \\ &= \max\left\{\operatorname{Re}\left[\sum_{n=1}^{\infty} a_n b_n\right] : f \in \mathfrak{H}\mathrm{St}^*\right\} = \max\left\{\operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{a_n}{n} n b_n\right] : f \in \mathfrak{H}\mathrm{St}^*\right\} \\ &= \max\left\{\operatorname{Re}\left[\sum_{n=1} c_n n b_n\right] : g \in \mathfrak{H}K^*\right\}, \end{aligned}$$

where $g(z) = \sum_{n=1}^{\infty} c_n z^n \in \mathfrak{H}K^*$. The last equality follows from the fact that $\mathfrak{L}(\mathfrak{H}\mathrm{St}^*) = \mathfrak{H}K^*$. Now, $G(z) = \sum_{n=1}^{\infty} n b_n z^n$ and, so,

$$\begin{aligned} \max\left\{\operatorname{Re}\left[\sum_{n=1}^{\infty} c_n n b_n\right] : g \in \mathfrak{H}K^*\right\} &= \max\{\operatorname{Re} J_G(g): g \in \mathfrak{H}K^*\} \\ &= \max\{\operatorname{Re} J_G(g): g \in K^*\}. \end{aligned}$$

This proves (27) and the remaining statements follow easily.

LEMMA 6. *Let F be an analytic nonconstant function on $\bar{\Delta}$ so that $|F(z)| \leq M$ for $|z| \leq 1$. If $|F(z)| = M$ for an infinite number of distinct values of z with $|z| = 1$, then F maps $\bar{\Delta}$ onto $\{w: |w| \leq M\}$, and $F = MG$ where G is a finite Blaschke product.*

PROOF. The conclusion that G is a finite Blaschke product follows from the previous assertion and the fact that finite Blaschke products are characterized by the conditions: G is analytic in Δ and continuous in $\bar{\Delta}$, and $|G(z)| = 1$ when $|z| = 1$.

Without loss of generality we may let $M = 1$. Let $A = \{F(e^{i\theta}): |F(e^{i\theta})| = 1\}$. If A is dense in $\{w: |w| = 1\}$ then the continuity of F on $\bar{\Delta}$ would complete the argument. Otherwise, there exists a point w' such that $|w'| = 1$ and F does not take on values in some open neighborhood of w'. The function $g(w) = (w - w')^{-1}$ is analytic on $F(\bar{\Delta})$ and maps $\{w: |w| = 1\}$ onto a straight line. Then $h = g \circ F$ is analytic in $\bar{\Delta}$, and $h(\partial\Delta)$ intersects a line an infinite number of times. It follows [**3**, p. 106] that h is constant. This provides the contradiction.

REMARK. Lemma 6, in particular, asserts that either $F(\partial\Delta) = \{w: |w| = M\}$ or $F(\partial\Delta) \cap \{w: |w| = M\}$ is a finite set.

LEMMA 7. *Suppose that $0 < \lambda_k < 1$ $(k = 1, 2, \ldots, m)$, $\sum_{k=1}^m \lambda_k = 1$, $|x_k| = |y_k| = 1$ $(k = 1, 2, \ldots, m)$ and the numbers y_k are distinct. The function*

$$f(z) = \sum_{k=1}^{m} \lambda_k \frac{x_k z}{1 - y_k z} \tag{28}$$

is in K^* if and only if y_k/x_k is constant $(k = 1, 2, \ldots, m)$.

PROOF. One direction is clear since $z/(1 - cz)$ belongs to K whenever $|c| = 1$, and $x_k z/(1 - c x_k z) \prec z/(1 - cz)$ for $k = 1, 2, \ldots, m$.

Now suppose that f is given by (28) and belongs to K^*, and so $f \prec F$, where $F \in K$. Since f has simple poles at $z = \bar{y}_k$ $(k = 1, 2, \ldots, m)$, it follows from [**4**, p. 87] that $F(z) = z/(1 - cz)$ for some c with $|c| = 1$.

The subordination $f(z) \prec z/(1 - cz)$ implies

$$\sum_{k=1}^{m} \lambda_k \frac{c x_k z}{1 - y_k z} \prec \frac{z}{1 - z}. \tag{29}$$

Since $z/(1 - z) = -\frac{1}{2} + \frac{1}{2}(1 + z)/(1 - z)$, (29) asserts that $g(z) = 1 + \sum_{k=1}^m 2\lambda_k c x_k z/(1 - y_k z)$ belongs to $\mathcal{P}$. Since g has a pole at $\bar{y}_k$ with residue $-2\lambda_k c x_k \bar{y}_k^2$, the first assertion in Lemma 3 implies that $2\lambda_k c x_k \bar{y}_k > 0$. From this we conclude that $c x_k \bar{y}_k = 1$, since $|c| = |x_k| = |y_k| = 1$. Therefore, $y_k = c x_k$ for $k = 1, 2, \ldots, m$.

THEOREM 11. *Suppose $F(z) = \sum_{n=0}^\infty b_n z^n$ is analytic in $\bar{\Delta}$ and $F - b_0$ does not map $\bar{\Delta}$ onto a disc centered at $w = 0$. Then there exist complex numbers x_k, y_k $(k = 1, 2, \ldots, m)$ such that $|x_k| = |y_k| = 1$, the numbers $\{y_k\}$ are distinct, and the support points of K^* associated with J_F are given by either*

$$\{x_k z/(1 - y_k z): k = 1, 2, \ldots, m\} \tag{30}$$

or

$$\left\{\sum_{k=1}^{m}\lambda_k\frac{x_k z}{1-y_k z}: 0<\lambda_k<1,\ \sum_{k=1}^{m}\lambda_k=1\right\}, \tag{31}$$

where y_k/x_k is constant ($k=1,2,\ldots,m$).

PROOF. Let $M=\max\{\operatorname{Re}J_F(f): f\in K^*\}$. Then

$$\begin{aligned} M &= \max\{\operatorname{Re}J_F(f): f\in \mathfrak{E}\mathcal{H}K^*\}\\ &= \max\left\{\operatorname{Re}\left[x\sum_{n=1}^{\infty}b_n y^{n-1}\right]: |x|=|y|=1\right\}\\ &= \max\{\operatorname{Re} xG(y): |x|=|y|=1\}\end{aligned}$$

where $G(z)=(F(z)-b_0)/z$. Since G does not map onto a disc centered at $w=0$, Lemma 6 implies that there exist only a finite number of points x_k, y_k ($k=1,2,\ldots,m$) so that $x_kG(y_k)=M$. The numbers y_k are distinct and are determined by $|G(y_k)|=M$, and the numbers x_k are determined by the condition $x_kG(y_k)>0$ (and $|x_k|=1$).

If $\mathcal{G}=\{f: f\in \mathfrak{H}K^* \text{ and } \operatorname{Re}J_F(f)=M\}$, then familiar arguments imply that $\mathcal{G}$ consists of the functions $f(z)=\sum_{k=1}^{m}\lambda_k x_k z/(1-y_k z)$, where $0\leqslant\lambda_k\leqslant 1$ and $\sum_{k=1}^{m}\lambda_k=1$. Since $x_kz/(1-y_kz)\in K^*$, Lemma 7 determines which functions in $\mathcal{G}$ also belong to K^* and the result is expressed through the assertions concerning (30) and (31).

REMARKS. 1. Theorem 11 holds whenever the functional J_F depends only on a finite number of coefficients of functions in $\mathcal{Q}$ and is not of the form $J(f)=\alpha f(0)+\beta f^{(n)}(0)$.

2. We recall that each function $xz/(1-yz)$ ($|x|=|y|=1$) belongs to supp K^* [7, p. 458] and so $\mathfrak{E}\mathfrak{H}K^*\subsetneq \operatorname{supp}K^*$.

LEMMA 8. *Suppose $F(z)=xz\prod_{k=1}^{m}(z+\alpha_k)/(1+\bar{\alpha}_k z)$ where $|\alpha_k|<1$ ($k=1,2,\ldots,m$) and $|x|=1$ and define $\bar{F}$ by $\bar{F}(z)=\overline{F(\bar{z})}$. Then $J_F(\bar{F}^n)=0$ for $n=2,3,\ldots$.*

PROOF. We note that $\bar{F}$ is analytic in $\bar{\Delta}$, $|F(z)|=1$ for $|z|=1$, and $\bar{F}(0)=0$. Thus,

$$\begin{aligned} J_F(\bar{F}^n) &= \frac{1}{2\pi}\int_0^{2\pi}[\bar{F}(e^{i\theta})]^n F(e^{-i\theta})\,d\theta \qquad [\mathbf{11}, \text{p. } 36]\\ &= \frac{1}{2\pi}\int_0^{2\pi}\overline{F(e^{-i\theta})}\,F(e^{-i\theta})[\bar{F}(e^{i\theta})]^{n-1}\,d\theta\\ &= \frac{1}{2\pi}|F(e^{-i\theta})|^2[\bar{F}(e^{i\theta})]^{n-1}\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}[\bar{F}(e^{i\theta})]^{n-1}\,d\theta = [\bar{F}(0)]^{n-1}=0\end{aligned}$$

for $n=2,3,\ldots$.

THEOREM 12. *Let F satisfy the conditions of Lemma* 8 *and let* $G = MF$ *where* $M > 0$. *If* $H \in K$ *then* $H(\bar{F})$ *is a support point of* K^* *associated with the continuous, linear functional* J_G.

PROOF. Suppose that $H(z) = z + \sum_{n=2}^{\infty} A_n z^n$ belongs to K. Then Lemma 8 implies that

$$J_G[H(\bar{F})] = J_G(\bar{F}) + \sum_{n=2}^{\infty} A_n J_G(\bar{F}^n) = J_G(\bar{F}) + \sum_{n=2}^{\infty} A_n M J_F(\bar{F}^n) = J_G(\bar{F}).$$

Since $\mathfrak{E}\mathfrak{H} K^* = \{xz/(1-yz): |x| = |y| = 1\}$, it is clear that

$$\max\{\operatorname{Re} J_G(g): g \in K^*\} = M.$$

Also

$$\begin{aligned} J_G(\bar{F}) &= \frac{1}{2\pi}\int_0^{2\pi} \bar{F}(e^{i\theta}) G(e^{-i\theta})\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{F(e^{-i\theta})}\, M F(e^{-i\theta})\, d\theta \\ &= \frac{M}{2\pi}\int_0^{2\pi} |F(e^{-i\theta})|^2\, d\theta = M. \end{aligned}$$

As $\operatorname{Re} J_G$ is not constant on K^*, this shows that $H(\bar{F})$ is a support point of K^*, since, for example, $\operatorname{Re} J_G(\bar{F}) = M > 0$ and $\operatorname{Re} J_G(\bar{F}^2) = 0$ and $\bar{F}^2 \prec z$ and, so, $\bar{F}^2 \in K^*$.

REMARKS. 1. Since $\bar{F}$ represents any finite Blaschke product vanishing at the origin, Theorem 12 implies the following result: If $G \in K$ and ϕ is a finite Blaschke product with $\phi(0) = 0$, then $G \circ \phi \in \operatorname{supp} K^*$.

2. Theorem 12 shows that for continuous, linear functionals of the form described, the set of support points is quite diverse. The support points of K^* in this theorem include the functions determined in (31) [**4**, p. 83].

COROLLARY 2. $\operatorname{supp} K^* = \{G \circ \phi$: $G \in K$, ϕ *is a finite Blaschke product*, $\phi(0) = 0\}$.

PROOF. This is a consequence of Theorems 4 and 12. In order to apply Theorem 4 a few observations are needed.

Suppose that $f \in \operatorname{supp} K^*$. Then there is a continuous, linear functional J on $\mathfrak{A}$ so that $\operatorname{Re} J$ is not constant on K^* and $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g)$: $g \in K^*\}$. Since $f \in K^*$, $f \prec F$ for some F in K. If $\mathfrak{F} = \{g$: $g \prec F$ in $\Delta\}$, then $f \in \mathfrak{F}$ and $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g)$: $g \in \mathfrak{F}\}$. To apply Theorem 4 it remains to show that $\operatorname{Re} J$ is not constant on $\mathfrak{F}$. This follows from the fact that the assertions: $\operatorname{Re} J$ is not constant on K^*, and $\operatorname{Re} J$ is not constant on $\mathfrak{F}$, are equivalent to: J does not have the form $J(g) = \alpha g(0)$ $(g \in \mathfrak{A})$. An argument is not difficult based on the information that $yz^n \in \mathfrak{F}$ whenever $|y| \leq \frac{1}{2}$ and $n = 1, 2, \ldots$ (recall that $\{w$: $|w| < \frac{1}{2}\} \subset F(\Delta)$).

THEOREM 13. *Suppose that* $M > 0$, $|\alpha_k| < 1$ $(k = 1, 2, \ldots, m)$ *and*

$$F(z) = Mz \prod_{k=1}^{m} \frac{z + \alpha_k}{1 + \bar{\alpha}_k z} = \sum_{n=1}^{\infty} b_n z^n.$$

Suppose that $G \in K$, G *does not have the form* $G(z) = z/(1 - cz)$ $(|c| = 1)$, *and* $\omega \in \mathfrak{B}_0$. *If* $G \circ \omega$ *is a support point of* K^* *associated with* J_F *then* $\omega = \bar{F}/M$.

PROOF. Theorem 12 and the argument given in the proof of Corollary 2 imply that $\mathrm{Re}\{J_F[\lambda G(\bar{F}/M) + (1 - \lambda)G(\omega)]\} = M$ whenever $0 < \lambda < 1$. Since $\lambda G(\bar{F}/M) + (1 - \lambda)G(\omega)$ is subordinate to G and also belongs to $\operatorname{supp} K^*$, there is a finite Blaschke product ϕ with $\phi(0) = 0$ so that

$$\lambda G(\bar{F}/M) + (1 - \lambda)G(\omega) = G(\phi). \tag{32}$$

Since G does not have the form $w = z/(1 - cz)$ $(|c| = 1)$ and ϕ is an inner function, by the argument given in the proof of Theorem 6 we find that (32) implies that $\omega = \phi = \bar{F}/M$.

REMARKS. 1. Theorems 11–13 provide a complete description of the connection between the support point and the functional for which it is extremal. The diversity of these functions for the functionals considered in Theorems 12 and 13 is remarkable. A classical example of the situation given in Theorem 13 is found by letting $F(z) = z^n$. This problem was treated by W. Rogosinski in [**10**, pp. 70–72]. Our result permits an easier description of the possible extremal functions determined by Rogosinski and shows that the particular problem he considered actually has all of the features which can occur in general.

2. Let $\mathfrak{F}$ be a compact subset of $\mathcal{A}$ and let $\mathfrak{F}^* = \{f: f \prec g \text{ for some } g \text{ in } \mathfrak{F}\}$. It is known [**9**, pp. 365–366] that $\mathfrak{E}\mathfrak{H}\mathfrak{F}^* \subset \{f \circ \phi: f \in \mathfrak{E}\mathfrak{H}\mathfrak{F}, \phi \in \mathfrak{B}_0\}$. Theorems 12 and 13 show that the analogous inclusion, with the consideration of support points replacing that of extreme points, is false. This follows since the functions

$$z \prod_{k=1}^{m} \frac{z + \alpha_k}{1 + \bar{\alpha}_k z} \qquad (|\alpha_k| < 1)$$

belong to $\operatorname{supp} K^*$ but are not subordinate to any function of the form $w = z/(1 - cz)$ $(|c| = 1)$.

The following lemmas shall be used in the discussion of the support points of St*.

LEMMA 9. *If* $\phi(z) = xz\prod_{k=1}^{m}(z + \alpha_k)/(1 + \bar{\alpha}_k z)$ *where* $|x| = 1$, $|\alpha_k| < 1$ $(k = 1, 2, \ldots, m)$ *and* $m \geq 1$, *then* $z\phi'(z)/\phi(z)$ *is real and greater than* 1 *for* $|z| = 1$. *Also,* $\min\{z\phi'(z)/\phi(z): |z| = 1\} = \mu > 1$.

PROOF. If $|z| = 1$ then $z = 1/\bar{z}$ and, thus,

$$\begin{aligned}\frac{z\phi'(z)}{\phi(z)} &= 1 + z \sum_{k=1}^{m} \left\{ \frac{1}{z + \alpha_k} - \frac{\bar{\alpha}_k}{1 + \bar{\alpha}_k z} \right\} \\ &= 1 + \sum_{k=1}^{m} \left(1 - |\alpha_k|^2\right) \frac{z}{(z + \alpha_k)(1 + \bar{\alpha}_k z)} \\ &= 1 + \sum_{k=1}^{m} \left(1 - |\alpha_k|^2\right) \frac{1}{|1 + \alpha_k z|^2} > 1.\end{aligned}$$

The second statement follows from the continuity of $z\phi'(z)/\phi(z)$ on $\partial\Delta$.

LEMMA 10. *Let* ϕ *be a finite Blaschke product with* $\phi(0) = 0$ *and not of the form* $\phi(z) = xz$ $(|x| = 1)$. *There is a number* $\mu > 1$ *so that if* $G \in K$, $H(z) = zG'(z)$, $L(z) = zG'(\phi(z))\phi'(z)$ *and* $w \in \partial L(\Delta)$, *then there is a real number* r' *and a point* ζ *so that* $\zeta \in \partial H(\Delta)$, $w = \zeta r'$ *and* $r' \geqslant \mu$.

PROOF. Let μ be given by Lemma 9 and assume that $w \in \partial L(\Delta)$. There exist points w_n in $L(\Delta)$ so that $w_n \to w$ and $w_n = L(z_n)$ where $|z_n| < 1$. Since L is not constant the open mapping theorem implies that $|z_n| \to 1$. We may select a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ so that $z_{n_k} \to z'$ as $k \to \infty$ and $|z'| = 1$. If $\phi(z) \neq 0$ then we may write

$$L(z) = H(\phi(z))r(z), \tag{33}$$

where $r(z) = z\phi'(z)/\phi(z)$. In particular, r is well defined and continuous on an open annulus containing $\partial\Delta$ and, consequently, $r(z_{n_k}) \to r(z')$ as $k \to \infty$. Also, $r(z_{n_k}) \neq 0$ for sufficiently large k since $r(z') \geqslant \mu$. The sequence $H(\phi(z_{n_k})) = L(z_{n_k})/r(z_{n_k})$ is defined for sufficiently large k and $H(\phi(z_{n_k})) \to w/r(z')$ as $k \to \infty$. Because $G \in K$, $H \in \mathrm{St}$ and so H is univalent in Δ. Therefore, $\zeta = \lim_{k\to\infty} H(\phi(z_{n_k}))$ must belong to $\partial H(\Delta)$ as $|\phi(z_{n_k})| \to |\phi(z')| = 1$. Hence, $w = \zeta r(z')$, which is the assertion.

LEMMA 11. *Suppose that* ϕ *is a finite Blaschke product*, $\phi(0) = 0$, $G \in K$ *and* $F \in S$. *If* $zG'(\phi(z))\phi'(z) \prec F(z)$ *in* Δ, *then* $\phi(z) = xz$ *and* $|x| = 1$.

PROOF. Let

$$H(z) = zG'(z) \quad \text{and} \quad L(z) = zG'(\phi(z))\phi'(z).$$

If $\alpha = \min\{|w| \colon w \in \partial H(\Delta)\}$, then, as $H \in S$, it is well known that $\frac{1}{4} \leqslant \alpha \leqslant 1$. Assume that ϕ satisfies the hypotheses of the lemma and does not have the form $\phi(z) = xz$ where $|x| = 1$. If $w \in \partial L(\Delta)$, then by Lemma 10 $w = \zeta r'$ where $\zeta \in \partial H(\Delta)$ and $r' \geqslant \mu$. Since $|w| \geqslant \alpha\mu$, $\mu > 1$ and $L(0) = 0$, we conclude that $L(\Delta)$ contains a disc centered at $w = 0$ and with radius larger than α. Thus, L cannot be subordinate to K in Δ.

We now show that $H(\Delta) \subset L(\Delta)$. Assume otherwise; then there exists a point w belonging to $\partial L(\Delta) \cap H(\Delta)$. By Lemma 10 there exists a point ζ belonging to $\partial H(\Delta)$ and lying between 0 and w on the line segment connecting these two points. This is a contradiction of the fact that H is a starlike mapping.

Since F is univalent, the relations $H(\Delta) \subset L(\Delta)$ and $L \prec F$ imply that $H(\Delta) \subset F(\Delta)$ and, hence, $H \prec F$. For two analytic functions H and F which satisfy $H(0) = F(0)$ and $|H'(0)| = |F'(0)|$, the subordination is possible only if $H(z) = F(yz)$ where $|y| = 1$. This relation between H and F implies a similar relation between H and L, which leads to the contradiction that $\phi(z) = xz$ with $|x| = 1$.

The next result originally was used to prove Theorem 14 in the case where F' does not map $\overline{\Delta}$ onto the disc $\{w \colon |w| \leqslant M\}$, but our present argument does not require this fact. This proposition, which has independent interest, is presented as a further application of Lemma 3.

PROPOSITION 1. *The only functions of the form* $f(z)=\sum_{k=1}^{m}\lambda_k x_k z/(1-y_k z)^2$, *where* $0\leqslant\lambda_k, |x_k|=|y_k|=1, \sum_{k=1}^{m}\lambda_k=1$ *and* $\{y_k\}$ *are distinct, which belong to* St* *have the form* $f(z)=xz/(1-yz)^2$ *where* $|x|=|y|=1$.

PROOF. If $f\in$ St* and has the stated form, then $f\prec F$ for some f in St and f has at least one pole of order 2. It follows from [4, p. 88] that $F(z)=z/(1-cz)^2$ and $|c|=1$. The subordination $f\prec F$ may be expressed

$$\sum_{k=1}^{m}\lambda_k\frac{cx_k z}{(1-y_k z)^2}\prec\frac{z}{(1-z)^2},$$

which is the same as

$$1+\sum_{k=1}^{m}4\lambda_k\frac{cx_k z}{(1-y_k z)^2}\prec\left(\frac{1+z}{1-z}\right)^2. \tag{34}$$

Since the right-hand side of (34) has no zeros in Δ, the left-hand side also has no zeros in Δ. Therefore, the square root of the left-hand side of (34) is analytic in Δ and subordinate to $(1+z)/(1-z)$, and so

$$\left[1+\sum_{k=1}^{m}4\lambda_k\frac{cx_k z}{(1-y_k z)^2}\right]^{1/2}=\int_X\frac{1+xz}{1-xz}\,d\mu(x), \tag{35}$$

where μ is a probability measure on $X=\partial\Delta$. The function on the left-hand side of (35) has poles at $\{\bar y_k\}$. If $\lambda_k>0$ then Lemma 3 implies that

$$2[cx_k/y_k]^{1/2}>0$$

and

$$\mu[\{\bar y_k\}]=-\frac{1}{2\bar y_k}\left[-2\lambda_k^{1/2}(cx_k)^{1/2}(\bar y_k)^{3/2}\right]=\left[\frac{\lambda_k cx_k}{y_k}\right]^{1/2}.$$

Therefore, $y_k=cx_k$ and $\mu[\{\bar y_k\}]=\lambda_k^{1/2}$. If $\lambda_k=1$ for some k or if $m=1$, then the result is clear. Otherwise, there are at least two values of k so that $\lambda_k>0$ and

$$\sum_{k=1}^{m}\mu[\{\bar y_k\}]=\sum_{k=1}^{m}\lambda_k^{1/2}>\sum_{k=1}^{m}\lambda_k=1,$$

which contradicts $\mu(X)=1$.

THEOREM 14. *Let* J *be a continuous, linear functional on* $\mathcal{Q}$. *If* J *does not have the form* $J(f)=\alpha f(0)+\beta f'(0)$, *then each support point of* St* *associated with* J *has the form* $xz/(1-yz)^2$ *where* $|x|=|y|=1$. *If* J *has the form* $J(f)=\alpha f(0)+\beta f'(0)$, *then there is a unique* x *with* $|x|=1$ *so that, for any* f *in* St, $f(xz)$ *is a support point of* St* *associated with* J.

PROOF. It is known that $\mathfrak{E}\mathfrak{H}$St* $\subset$ supp St* [7, p. 459]. Let f_0 be a support point of St* associated with the functional J_F where $F(z)=\sum_{n=0}^{\infty}b_n z^n\in\mathcal{Q}$, and let $E(z)=zF'(z)$. It follows that $\operatorname{Re}J_F$ is not constant on $\mathfrak{E}\mathfrak{H}$St* and so $\operatorname{Re}J_E$ is not constant on $\mathfrak{E}\mathfrak{H}K^*$. This implies that $\operatorname{Re}J_E$ is not constant on K^*. By Lemma 5, $g_0(z)=\int_0^z f_0(\zeta)/\zeta\,d\zeta$ is a support point of K^* associated with the functional J_E. Corollary 2 implies that $g_0 = G \circ \phi$, where $G \in K$ and ϕ is a finite Blaschke product with $\phi(0) = 0$. Since $f_0(z) = zg_0'(z) = zG'(\phi(z))\phi'(z)$ is in St*, Lemma 11 shows that $\phi(z) = vz$ and $|v| = 1$. Thus, $G(vz)$ is a support point of K^* associated with the functional J_E. Since $f(xz) \in K^*$ whenever $f \in K^*$ and $|x| = 1$, we see that G is a support point of K^* associated with the functional J_H, where $H(z) = \sum_{n=1}^{\infty} nb_n v^n z^n$. The relations $G \in K$ and $G \in \operatorname{supp} K^*$ imply that $G \in \operatorname{supp} K$ if we prove that $\operatorname{Re} J_H$ is not constant on K. Assume that $\operatorname{Re} J_H$ is constant on K. We recall that for each complex number x with $|x| = 1$ and each integer $n \geq 2$, the function $z + xz^n/n^2$ is in K. Since $J_H(z + xz^n/n^2) = b_1 v + xb_n v^n/n$, we conclude that $b_n = 0$ for $n = 2, 3, \ldots$. This implies that $J(f) = b_0 f(0) + b_1 f'(0)$ and, so, in the case J does not have the form $J(f) = \alpha f(0) + \beta f'(0)$, we find that $G \in \operatorname{supp} K$. Therefore [**3**, p. 102], $G(z) = z/(1 - cz)$ where $|c| = 1$. Then $g_0(z) = vz/(1 - cvz)$ and $f_0(z) = vz/(1 - cvz)^2$.

In the case $F(z) = b_0 + b_1 z$, we argue as follows. If $f(z) = \sum_{n=1}^{\infty} a_n z^n \in$ St*, then there is a function g in St and a function ϕ in $\mathfrak{B}_0$ so that $f = g \circ \phi$. Schwarz's lemma implies that $|a_1| \leq 1$ and $|a_1| = 1$ if and only if $\phi(z) = xz$ where $|x| = 1$. Clearly there is a unique value of x so that $xb_1 = |b_1|$ and $|x| = 1$. Thus $f(xz)$ is a support point of St* associated with J_F.

Remarks. 1. If $F'(z)$ does not map $\overline{\Delta}$ onto the disc $\{w: |w| \leq M\}$, where $M = \max\{\operatorname{Re} J_F(f): f \in \text{St}^*\}$, then arguments based on Lemma 6 and Proposition 1 show that the solution set over St* contains a finite number of extreme points of $\mathfrak{H}$St*. If $F'(z)$ maps $\overline{\Delta}$ onto the disc above, then the solution set over St* contains an infinite number of members of $\mathfrak{E}\mathfrak{H}$St*.

2. Let C denote the subset of S consisting of close-to-convex functions, and let $C^* = \{f: f \prec g \text{ for some } g \text{ in } C\}$. In [**6**] it was proven that

$$\mathfrak{E}\mathfrak{H}C^* \subset \left\{ w\frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2} : |x| = |y| = |w| = 1, x \neq y \right\}.$$

This inclusion can be shown to be an equality by an argument similar to that used in [**3**, p. 98]. This leads to the problem of determining $\operatorname{supp} C^*$.

Added in proof. S. Porera and D. Wilken have found more direct proofs of Lemma 4 and Theorem 10. They also have an easy proof of the assertion made above in Remark 2.

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